2.30 $F(x,y) = (x+1)^2 + y^2x^3$

(a) Find $F_y(-2,3)$

$$F_y = 2yx^3 = 2(3)(-2)^3 = \boxed{-48}.$$

(b) Unit vector in $3\hat{i} + 4\hat{j}$ direction.

$$u = \frac{1}{\sqrt{9+16}}(3,4) = \left(\frac{3}{5}, \frac{4}{5}\right)$$

(c) $F_u(-2,3) = ?$

$$\frac{\partial F}{\partial u} = \nabla F \cdot u$$

$$F_x = 2(x+1) + 3y^2x^2 \overset{(-2,3)}{=} \boxed{106}$$

$$\frac{\partial F}{\partial u} = \nabla F \cdot u = (106, -48) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = \ldots \frac{126}{5}$$

(d) Crit. pts $\nabla F = (0,0)$

$$2(x+1) + 3y^2x^2 = 0$$

$$2yx^3 = 0$$

$y = 0$ or $x = 0$

$2(x+1) + 0 = 0$

$x = -1$

$(-1, 0)$

$2(0+1) + 0 = 0$

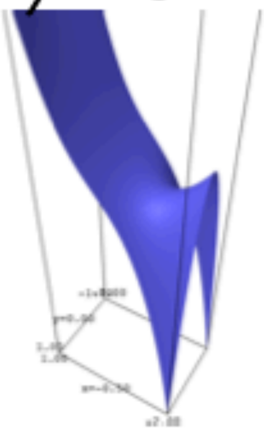
$2 = 0$

impossible

$\Rightarrow$ only 1 cr. pt.

$$\boxed{(-1, 0)}$$

(e) Type of crit pt by looking at graph.



$(-1,0)$ is a Saddle pt!

(f) At each ~~each~~ critical pt, find $\frac{\partial F}{\partial v}$, where $v = \left(\frac{5}{13}, \frac{-12}{13}\right)$.

$$\frac{\partial F}{\partial v} = \nabla F \cdot v = (0,0) \cdot v = \boxed{0}$$

Back to our example:

$$h(x,y) = \left(\tfrac{1}{2} - x^2 + y^2\right) e^{1-x^2-y^2}$$

$$\nabla h = \left(-2x\left(\tfrac{3}{2} - x^2 + y^2\right) e^{1-x^2-y^2},\ 2y\left(\tfrac{1}{2} + x^2 - y^2\right) e^{1-x^2-y^2} \right)$$

We solved for $\nabla h = (0,0)$ to get the critical points:

$(0,0)$, $\left(0, \frac{1}{\sqrt{2}}\right)$, $\left(0, \frac{-1}{\sqrt{2}}\right)$, $\left(\sqrt{\frac{3}{2}}, 0\right)$, $\left(-\sqrt{\frac{3}{2}}, 0\right)$.

To find the type of each of these, we need to calculate the Hessian matrix.

We have
$$h_x = -2x\left(\tfrac{3}{2}-x^2+y^2\right)e^{1-x^2-y^2}$$
$$h_y = 2y\left(\tfrac{1}{2}+x^2-y^2\right)e^{1-x^2-y^2}$$

Then
$$\begin{aligned} h_{xx} = &-2\left(\tfrac{3}{2}-x^2+y^2\right)e^{1-x^2-y^2} \\ &+(-2x)(-2x)e^{1-x^2-y^2} \\ &+(-2x)\left(\tfrac{3}{2}-x^2+y^2\right)e^{1-x^2-y^2}\cdot(-2x) \end{aligned}$$

$$(fgh)' = f'gh + fg'h + fgh'$$

From sage math,

$$H = \begin{pmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{pmatrix}$$

$$\begin{pmatrix} -(4x^4 - 4x^2y^2 - 12x^2 + 2y^2 + 3)e^{(-x^2-y^2+1)} & -2\,(2x^2 - 2y^2 - 1)xye^{(-x^2-y^2+1)} \\ -2\,(2x^2 - 2y^2 - 1)xye^{(-x^2-y^2+1)} & -(4x^2y^2 - 4y^4 - 2x^2 + 8y^2 - 1)e^{(-x^2-y^2+1)} \end{pmatrix}$$

So now we need to evaluate the matrix at each critical point.

$$(0,0),\ \left(0,\tfrac{1}{\sqrt{2}}\right),\ \left(0,-\tfrac{1}{\sqrt{2}}\right),\ \left(\sqrt{\tfrac{3}{2}},0\right),\ \left(-\sqrt{\tfrac{3}{2}},0\right).$$

For $(0,0)$,

$$H = \begin{pmatrix} -3e^1 & 0 \\ 0 & +1e^1 \end{pmatrix}$$

eigenvalues: $\det(H - \lambda I) = 0$, solve for λ

$$\det\begin{pmatrix} -3e^1-\lambda & 0 \\ 0 & e-\lambda \end{pmatrix} = (-3e-\lambda)(e-\lambda)=0$$

$$\Rightarrow \boxed{\lambda = \underset{\text{neg}}{-3e} \text{ or } \lambda = \underset{\text{pos.}}{e}.}$$

This must be a saddle pt.

Now let's look at the critical pts $\left(0, \pm\frac{1}{\sqrt{2}}\right)$

$$-\left(4x^4-4x^2y^2-12x^2+2y^2+3\right)e^{(-x^2-y^2+1)}$$

$$= -\left(2\left(\tfrac{1}{2}\right)+3\right)e^{-\frac{1}{2}+1} = -4e^{1/2}$$

$$-2\left(2x^2-2y^2-1\right)xye^{\left(-x^2-y^2+1\right)}$$

$$= -2\left(-2\left(\tfrac{1}{2}\right)-1\right)0 = 0$$

$$-\left(4x^2y^2-4y^4-2x^2+8y^2-1\right)e^{\left(-x^2-y^2+1\right)}$$

$$= -\left(-4\left(\tfrac{1}{4}\right)+8\left(\tfrac{1}{2}\right)-1\right)e^{-\frac{1}{2}+1}$$

$$= -\left(-1+4-1\right)e^{1/2} = -2e^{1/2}$$

for $\left(0, \pm\frac{1}{\sqrt{2}}\right)$,

$$H = \begin{pmatrix} -4e^{1/2} & 0 \\ 0 & -2e^{1/2} \end{pmatrix}$$ eigenvalues $= -4e^{1/2}, -2e^{1/2}$

$\Rightarrow$ These are both local maxes.

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last critical points: $\left(\sqrt{\frac{3}{2}}, 0\right), \left(-\sqrt{\frac{3}{2}}, 0\right)$

$$-\left(4x^4-4x^2y^2-12x^2+2y^2+3\right)e^{\left(-x^2-y^2+1\right)}$$

$$= -\left(4\left(\tfrac{9}{4}\right)-12\left(\tfrac{3}{2}\right)+3\right)e^{-\frac{3}{2}+1}$$

$$= -(9-18+3)e^{-1/2} = 6e^{-1/2}$$

Note $h_{xy} = h_{yx} = 0$.

$h_{yy} =$

$$-(4x^2y^2 - 4y^4 - 2x^2 + 8y^2 - 1)e^{(-x^2-y^2+1)}$$

$$= -\big(\underbrace{-2\left(\tfrac{3}{2}\right) - 1}_{-4}\big)e^{-\frac{3}{2}+1} = 4e^{-1/2}$$

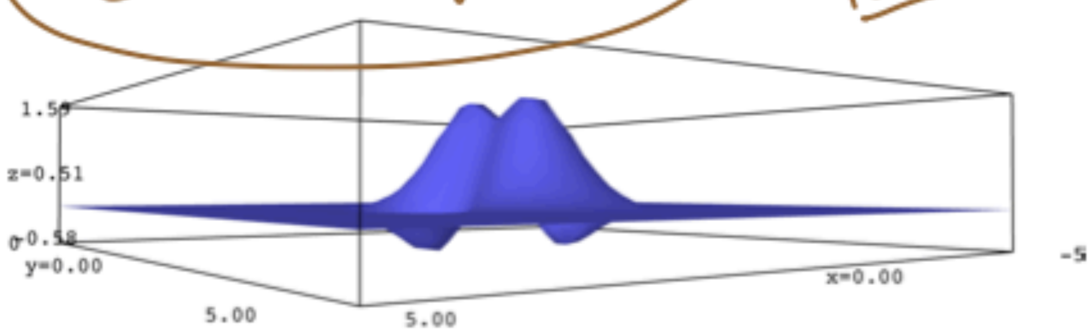
$$H = \begin{pmatrix} 6e^{-1/2} & 0 \\ 0 & 4e^{-1/2} \end{pmatrix}$$

eigenvalues $\lambda = 4e^{-1/2}, 6e^{-1/2}$.

So $\left(\pm\sqrt{\tfrac{3}{2}}, 0\right)$ are local mins.

Check on Sagemath

looks good!



Next topic — Optimization

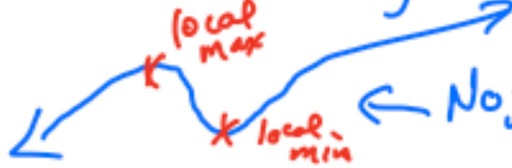
— find the global (absolute) maximum or minimum of a function of several variables, usually with constraints on the domain.

For the last fcn we examined,

$$h(x,y) = \left(\tfrac{1}{2} - x^2 + y^2\right) e^{-x^2-y^2+1},$$

we found local maxima & minima, but we could also tell by looking at the function that $h(x,y) \to 0$ as (x,y) moves away from the origin. So if this fcn has a maximum or minimum, it has to be a critical point ⇝ in this case we found the global max & min.

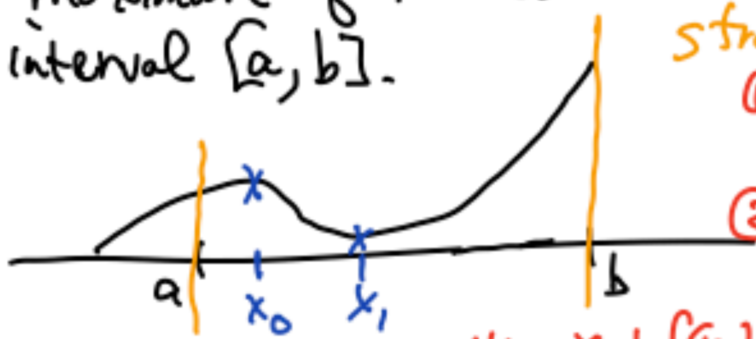
But there are many other types of situations



← No global max or min.

Reminder of how we deal with this in 1-variable calculus:

<u>Example</u>: $y = f(x)$ — find the maximum of f restricted to an interval $[a, b]$.



Strategy:

① Solve $f' = 0$ possible interior max/min.

② Check boundary pts. $x = a$, $x = b$

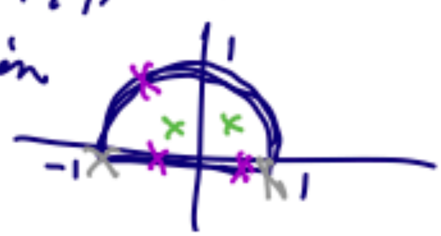
possible x	$f(x)$	
x_0	$f(x_0) = ---$	
x_1	$f(x_1) = $ ___	← abs min.
a	$f(a) =$	
b	$f(b) =$	← abs. max

b is called the absolute max

$f(b)$ is called the abs. max. value.

We will exactly the same for optimization for functions of several variables.

eg. maximize $g(x,y)$ when (x,y) is in this region



Strategy:

① Look for interior critical pts.
set $\nabla g = (0,0)$ $\leftarrow$ only consider those found inside.

② Look for boundary critical pts.
Consider the fcn restricted to the boundary.

③ Look at boundaries of boundary pieces.